Recent Extensions of the Discontinuous Enrichment Method for Variable-Coefficient Advection-Diffusion Problems in the High Péclet Regime

Irina Kalashnikova[1], R. Tezaur[2], C. Farhat[1,2]
[1] Institute for Computational & Mathematical Engineering (iCME)
[2] Department of Aeronautics & Astronautics
Stanford University

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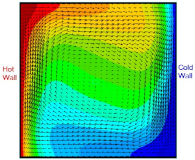
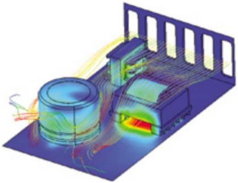
Scalar Advection-Diffusion Equation

$$\mathcal{L}c = \underbrace{-\kappa \Delta c}_{\text{diffusion}} + \underbrace{\mathbf{a} \cdot \nabla c}_{\text{advection}} = f$$

- 2D advection velocity vector: $\mathbf{a} = (a_1, a_2)^T = |\mathbf{a}|(\cos\phi, \sin\phi)^T$.
- ϕ = advection direction.
- κ = diffusivity.





- Describes many transport phenomena in fluid mechanics:
  - Heat transfer.
  - Semi-conductor device modeling.
  - Usual scalar model for the more challenging Navier-Stokes equations.
- Global **Péclet number** (L = length scale associated with Ω):

$$\boxed{Pe = \frac{\text{rate of advection}}{\text{rate of diffusion}} = \frac{L|\mathbf{a}|}{\kappa} = Re \cdot \begin{cases} Pr & \text{(thermal diffusion)} \\ Sc & \text{(mass diffusion)} \end{cases}}$$

Advection-Dominated Regime

- Typical applications: flow is advection dominated.

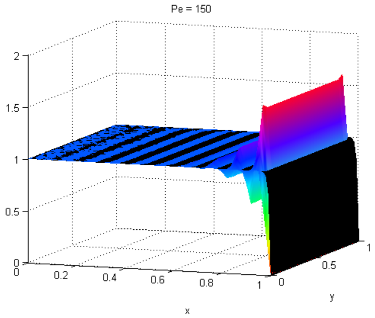


Figure 1: Galerkin Q_1 solution (color) vs. exact solution (black) ($Pe = 150$)

Advection-Dominated (High Pe) Regime
$\Downarrow$
Sharp gradients in exact solution
$\Downarrow$
Galerkin FEM inadequate: spurious oscillations (Fig. 1)

- Some classical remedies:
  - **Stabilized FEMs** (SUPG, GLS, USFEM): add weighted residual (numerical diffusion) to variational equation.
  - **RFB, VMS, PUM**: construct conforming spaces that incorporate knowledge of local behavior of solution.

The Discontinuous Enrichment Method (DEM)

- First developed by Farhat *et. al.* in 2000 for the Helmholtz equation [1].

Idea of DEM:

"Enrich" the usual Galerkin polynomial field $\mathcal{V}^P$ by the free-space solutions to the governing homogeneous PDE $\mathcal{L}c = 0$.

$$c^h = c^P + c^E \in \mathcal{V}^P \oplus (\mathcal{V}^E \backslash \mathcal{V}^P)$$

where

$$\mathcal{V}^E = \text{span}\{c : \mathcal{L}c = 0\}$$

- **Simple 1D Example:**

$$\begin{cases} u_x - u_{xx} = 1 + x, & x \in (0,1) \\ u(0) = 0, u(1) = 1 \end{cases}$$

  - *Enrichments*: $u_x^E - u_{xx}^E = 0 \Rightarrow u^E = C_1 + C_2 e^x \Rightarrow \mathcal{V}^E = \text{span}\{1, e^x\}$.
  - *Galerkin FEM polynomials*: $\mathcal{V}^P_{\Omega^e = (x_j, x_{j+1})} = \text{span}\left\{\frac{x_{j+1} - x}{h}, \frac{x - x_j}{h}\right\}$.

Two Variants of DEM

- Two variants of DEM: **"pure DGM"** vs. **"true DEM"**

	DGM	DEM
$\mathcal{V}^h$	$\mathcal{V}^E$	$\mathcal{V}^P \oplus (\mathcal{V}^E \backslash \mathcal{V}^P)$
c^h	c^E	$c^P + c^E$

Enrichment-Only "Pure DGM":
The standard continuous Galerkin polynomial field is dropped entirely from the approximation.

True or "Full" DEM:
Splitting of the approximation into coarse (polynomial) and fine (enrichment) scales.

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- Unlike PUM, VMS & RFB: enrichment field in DEM is not required to vanish at element boundaries $\Rightarrow$ DEM is **discontinuous** by construction!

DEM = DGM with Lagrange Multipliers

What about Inter-Element Continuity?

- Continuity across element boundaries is enforced weakly using Lagrange multipliers $\lambda^h \in \mathcal{W}^h$:

$$\boxed{\begin{array}{c}\lambda^h \approx \nabla c_e^E \cdot \mathbf{n}^e = -\nabla c_{e'}^E \cdot \mathbf{n}^{e'} \qquad \text{on } \Gamma^{e,e'} \\ \textit{but making sure we uphold the...}\end{array}}$$

- Discrete **Babuška-Brezzi *inf-sup* condition**[1]:

$$\left\{ \begin{array}{c}\text{\# Lagrange multiplier} \\ \text{constraint equations}\end{array} \leq \begin{array}{c}\text{\# enrichment} \\ \text{equations}\end{array} \right\}$$

Rule of thumb to satisfy the Babuška-Brezzi *inf-sup* condition is to limit:

$$n^\lambda = \left\lfloor \frac{n^E}{4} \right\rfloor \equiv \max \left\{ n \in \mathbb{Z} | n \leq \frac{n^E}{4} \right\}$$

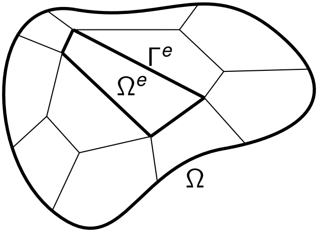
$$\begin{aligned} n^\lambda &= \text{\# Lagrange multipliers per edge} \\ n^E &= \text{\# enrichment functions} \end{aligned}$$

[1] Necessary condition for generating a non-singular global discrete problem.

Hybrid Variational Formulation of DEM

- Strong form:

$$(S):\left\{\begin{array}{rcll} \multicolumn{4}{l}{\text{Find } c \in H^1(\Omega) \text{ such that}} \\ -\kappa\Delta c + \mathbf{a}\cdot\nabla c & = & f, & \text{in } \Omega \\ c & = & g, & \text{on } \Gamma = \partial\Omega \end{array}\right.$$



Notation:

$\tilde{\Omega} = \cup_{e=1}^{n_{el}} \Omega^e$

$\tilde{\Gamma} = \cup_{e=1}^{n_{el}} \Gamma^e$

$\Gamma^{e,e'} = \Gamma^e \cap \Gamma^{e'}$

$\Gamma^{\text{int}} = \cup_{e'<e} \cup_{e=1}^{n_{el}} \{\Gamma^e \cap \Gamma^{e'}\}$

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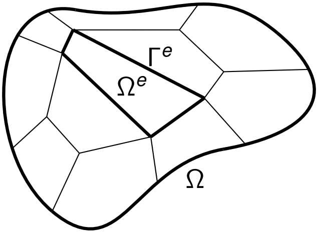

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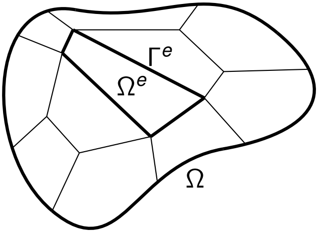

- Weak hybrid variational form:

$$(W): \begin{cases} \text{Find } (c,\lambda) \in \mathcal{V}\times\mathcal{W} \text{ such that:} & & & \\ a(v,c) & + & b(\lambda,v) & = & r(v) \\ b(\mu,c) & & & = & -r_d(\mu) \\ \text{holds } \forall c \in \mathcal{V}, \forall\mu \in \mathcal{W}. & & & & \end{cases}$$

where

$$a(v,c) = (\kappa\nabla v + v\mathbf{a}, \nabla c)_{\tilde{\Omega}}$$

$$b(\lambda,v) = \sum_e \sum_{e'<e} \int_{\Gamma^{e,e'}} \lambda(v_{e'} - v_e)d\Gamma + \int_\Gamma \lambda v\, d\Gamma$$

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Discretization & Implementation

- Element matrix problem (uncondensed):

$$\begin{pmatrix} \mathbf{k}^{\mathrm{PP}} & \mathbf{k}^{\mathrm{PE}} & \mathbf{k}^{\mathrm{PC}} \\ \mathbf{k}^{\mathrm{EP}} & \mathbf{k}^{\mathrm{EE}} & \mathbf{k}^{\mathrm{EC}} \\ \mathbf{k}^{\mathrm{CP}} & \mathbf{k}^{\mathrm{CE}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{c}^{\mathrm{P}} \\ \mathbf{c}^{\mathrm{E}} \\ \lambda^{h} \end{pmatrix} = \begin{pmatrix} \mathbf{r}^{\mathrm{P}} \\ \mathbf{r}^{\mathrm{E}} \\ \mathbf{r}^{\mathrm{C}} \end{pmatrix}$$

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- Statically-condensed **True DEM Element**:

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- Statically-condensed **Pure DGM Element**:

$$-\mathbf{k}^{\mathrm{CE}}(\mathbf{k}^{\mathrm{EE}})^{-1}\mathbf{k}^{\mathrm{EC}}\boldsymbol{\lambda}^{h} = \mathbf{r}^{C} - \mathbf{k}^{\mathrm{CE}}(\mathbf{k}^{\mathrm{EE}})^{-1}\mathbf{r}^{\mathrm{E}}$$

Discretization & Implementation

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Computational complexity depends on $\dim \mathcal{W}^h$ not on $\dim \mathcal{V}^E$

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Polynomial Enrichment Functions for 2D Advection-Diffusion

- Polynomial free-space solutions to $\mathbf{a} \cdot \nabla c_n^E - \Delta c_n^E = 0$ (of any desired degree n) can be derived[2].

[2] The advantage of these polynomials over standard FEM polynomials is discussed in: S. Brogniez, C. Farhat, "Theoretical Analysis of the DEM for the Advection-Diffusion Equation at High *Pe* Number," *FEF 2011* (W2G, Wed. March 23, 14:10-14:30).

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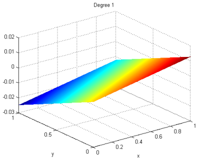

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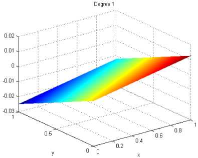

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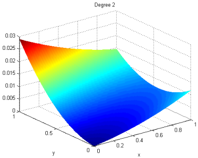

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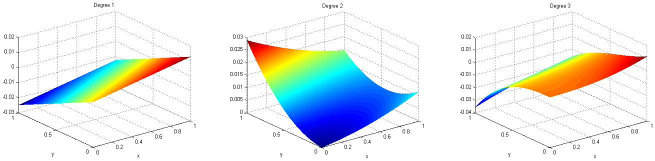
$$c_3^E(\mathbf{x}) = |\mathbf{a} \times \mathbf{x}|^3 + 6|\mathbf{a} \times \mathbf{x}|(\mathbf{a} \cdot \mathbf{x})$$

$$\vdots$$



$c_1^E(\mathbf{x})$ $c_2^E(\mathbf{x})$ $c_3^E(\mathbf{x})$

Slowly-varying (coarse) scale shape functions

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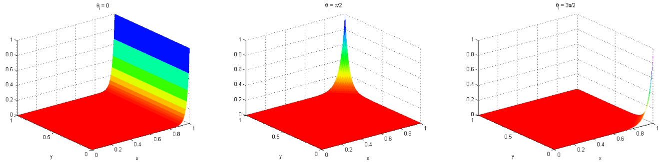
Angle-Parametrized Exponential Enrichment Functions for 2D Advection-Diffusion

- Exponential free-space solutions to $\mathbf{a} \cdot \nabla c^E - \kappa \Delta c^E = 0$ can be derived as well.

$$\boxed{c^E(\mathbf{x}; \theta_i) = e^{\left(\frac{a_1 + |\mathbf{a}| \cos \theta_i}{2\kappa}\right)(x - x_{r,i})} e^{\left(\frac{a_2 + |\mathbf{a}| \sin \theta_i}{2\kappa}\right)(y - y_{r,i})}} \quad (1)$$

$$\Theta^c \equiv \{\theta_i\}_{i=1}^{n^E} \in [0, 2\pi) = \text{ set of angles specifying } \mathcal{V}^E$$



$\phi = 0,\ \theta_i = 0$ $\qquad$ $\phi = 0,\ \theta_i = \frac{\pi}{2}$ $\qquad$ $\phi = 0,\ \theta_i = \frac{3\pi}{2}$

Figure 2: Plots of enrichment functions $c^E(\mathbf{x}; \theta_i)$ for several values of θ_i ($Pe = 20$)

Rapidly-varying (fine) scale shape functions

Lagrange Multiplier Approximations

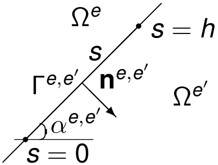


Figure 3: Straight edge $\Gamma^{e,e'}$ oriented at angle $\alpha^{e,e'} \in [0, 2\pi)$

$$\boxed{\lambda^h \approx \nabla c_e^E \cdot \mathbf{n}^e = -\nabla c_{e'}^E \cdot \mathbf{n}^{e'}}$$

Limit n^λ to satisfy *inf-sup*:

Use $\left\{ \begin{array}{ll} \left\lfloor \frac{n^{\text{exp}}}{4} \right\rfloor & \text{exponential LMs} \\ \left\lfloor \frac{n^{\text{pol}}}{4} \right\rfloor & \text{polynomial LMs} \end{array} \right.$

- LM approximations arising from exponential enrichments:

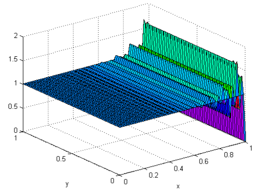
$$\lambda^h|_{\Gamma^{e,e'}} = \text{span}\left\{ e^{\Lambda_i^{e,e'}(s - s_{r,i}^{e,e'})},\ 0 \le s \le h, 1 \le i \le n^{\text{exp}} \right\}$$

where $\Lambda_i^{e,e'} \equiv \frac{|\mathbf{a}|}{2\kappa}\left[\cos(\phi - \alpha^{e,e'}) + \cos(\theta_i - \alpha^{e,e'})\right]$.

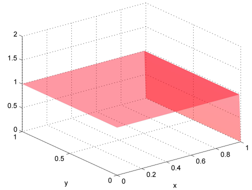
- LM approximations arising from polynomial enrichments:

$$\lambda^h|_{\Gamma^{e,e'}} = \text{span}\left\{ s^k,\ 0 \le s \le h, 0 \le k \le n^{\text{pol}} - 1 \right\}$$

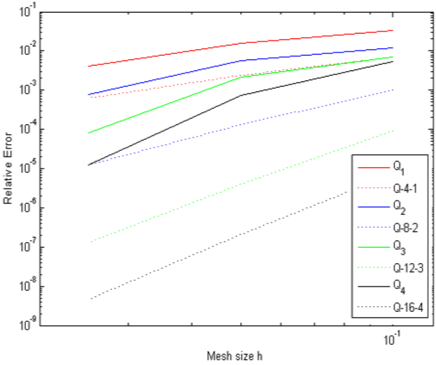
Representative Numerical Results for Constant-Coefficient Homogeneous Problem



Galerkin



DGM



To achieve relative error of 0.1%:

4.5 × fewer dofs

15 × fewer dofs

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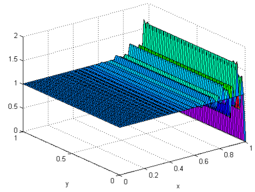

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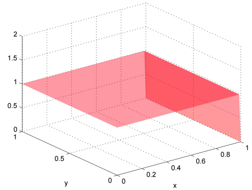
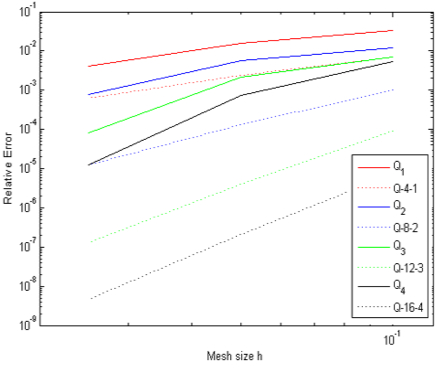


DGM



To achieve relative error of 0.1%:

$\approx 8 \times$ *less CPU time*

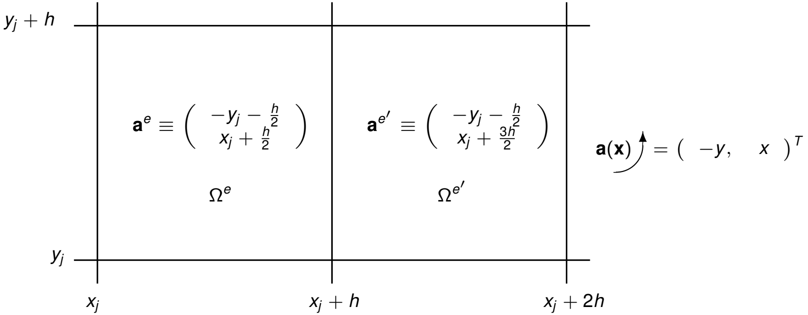
$\approx 40 \times$ *less CPU time*

Impressive results for constant-coefficient problems!

Extension to Variable-Coefficient Problems

- Define $\mathcal{V}^E$ *within each element* as the free-space solutions to the homogeneous PDE, with locally-*frozen* coefficients.
- $\mathbf{a}(\mathbf{x}) \approx \mathbf{a}^e$ =constant inside each element Ω^e as $h \to 0$:

$$\{\mathbf{a}(\mathbf{x}) \cdot \nabla c - \kappa \Delta c = f(\mathbf{x}) \ \text{ in } \ \Omega\} \approx \cup_{e=1}^{n_{el}} \{\mathbf{a}^e \cdot \nabla c - \kappa \Delta c = f(\mathbf{x}) \ \text{ in } \ \Omega^e\}.$$



- Enrichment in each element:

$$c_e^E(\mathbf{x};\theta_i^e) = e^{\frac{|\mathbf{a}^e|}{2\kappa}(\cos\phi^e + \cos\theta_i^e)(x - x_{r,i}^e)} e^{\frac{|\mathbf{a}^e|}{2\kappa}(\sin\phi^e + \sin\theta_i^e)(y - y_{r,i}^e)} \in \mathcal{V}_e^E$$

Relation Between Local Enrichment and Governing PDE

- Given $\mathbf{a}(\mathbf{x}) \in C^1(\Omega^e)$, Taylor expand $\mathbf{a}(\mathbf{x})$ around an element's midpoint $\bar{\mathbf{x}}^e$:

$$\mathbf{a}(\mathbf{x}) = \mathbf{a}(\bar{\mathbf{x}}^e) + \nabla \mathbf{a}|_{\mathbf{x}=\bar{\mathbf{x}}^e} \cdot (\mathbf{x} - \bar{\mathbf{x}}^e) + \mathcal{O}(\mathbf{x} - \bar{\mathbf{x}}^e)^2 \qquad \text{in } \Omega^e$$

Relation Between Local Enrichment and Governing PDE

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$$\mathbf{a}(\mathbf{x}) = \mathbf{a}(\bar{\mathbf{x}}^e) + \nabla \mathbf{a}|_{\mathbf{x}=\bar{\mathbf{x}}^e} \cdot (\mathbf{x} - \bar{\mathbf{x}}^e) + \mathcal{O}(\mathbf{x} - \bar{\mathbf{x}}^e)^2 \quad \text{in } \Omega^e$$

- Operator governing the PDE inside the element Ω^e takes the form

$$\boxed{\mathbf{a}(\mathbf{x})} \cdot \nabla c - \kappa \Delta c = \mathcal{L}_e c + f(c) \quad \text{in } \Omega^e$$

where

$$\mathcal{L}_e c \equiv \mathbf{a}(\bar{\mathbf{x}}^e) \cdot \nabla c - \kappa \Delta c$$

$$f(c) \equiv \left[\nabla \mathbf{a}|_{\mathbf{x}=\bar{\mathbf{x}}^e} \cdot (\mathbf{x} - \bar{\mathbf{x}}^e) + \mathcal{O}(\mathbf{x} - \bar{\mathbf{x}}^e)^2 \right] \cdot \nabla c$$

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Can we improve the enrichment space for variable-coefficient problems?

"Higher Order" Enrichment Function for Variable-Coefficient Advection-Diffusion

- Linearize $\mathbf{a}(\mathbf{x})$ to second order, instead of to first order:

$$\mathbf{a}(\mathbf{x}) \approx \mathbf{a}(\bar{\mathbf{x}}^e) + \nabla \mathbf{a}|_{\mathbf{x}=\bar{\mathbf{x}}^e} \cdot (\mathbf{x} - \bar{\mathbf{x}}^e) \qquad \text{in } \Omega^e$$

- Enrich with free-space solutions to

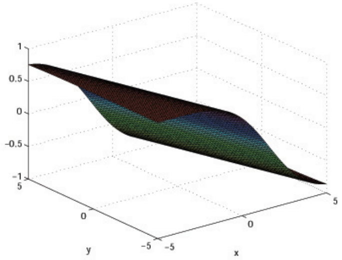
$$[\mathbf{A}\mathbf{x} + \mathbf{b}] \cdot \nabla c^E - \kappa \Delta c^E = 0 \tag{2}$$

where $\mathbf{A} \equiv \nabla \mathbf{a}|_{\mathbf{x}=\bar{\mathbf{x}}^e}$, $\mathbf{b} \equiv (\mathbf{a}(\bar{\mathbf{x}}^e) - \nabla \mathbf{a}|_{\mathbf{x}=\bar{\mathbf{x}}^e} \bar{\mathbf{x}}^e)$.

- Solutions to (2) are given by:

$$\boxed{c^E(\mathbf{x}) = \int_0^{\mathbf{v}_i \cdot \mathbf{x}} \exp\left\{ \frac{\sigma_i w^2}{2} + (\mathbf{v}_i \cdot \mathbf{b}) w \right\} dw}$$

$$\sigma_i = \text{eigenvalue of } \nabla \mathbf{a}|_{\mathbf{x}=\bar{\mathbf{x}}^e}$$
$$\mathbf{v}_i = \text{eigenvector of } \nabla \mathbf{a}|_{\mathbf{x}=\bar{\mathbf{x}}^e}$$



"Enrichment Function Bank"

Polynomial Family

$$c^E_{e,0}(\mathbf{x}) = 1$$
$$c^E_{e,1}(\mathbf{x}) = |\mathbf{a}^e \times \mathbf{x}|$$
$$c^E_{e,2}(\mathbf{x}) = |\mathbf{a}^e \times \mathbf{x}|^2 + 2(\mathbf{a}^e \cdot \mathbf{x})$$
$$c^E_{e,3}(\mathbf{x}) = |\mathbf{a}^e \times \mathbf{x}|^3 + 6|\mathbf{a}^e \times \mathbf{x}|(\mathbf{a}^e \cdot \mathbf{x})$$
$$\vdots$$

$\longrightarrow \mathcal{V}^E_e$

"Enrichment Function Bank"

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$$\vdots$$

$$\to \mathcal{V}^E_e$$

Exponential Family

$$c^E_e(\mathbf{x};\theta_i) = e^{\left(\frac{a^e_1 + |\mathbf{a}^e|\cos\theta_i}{2\kappa}\right)(x - x_{r,i})} e^{\left(\frac{a^e_2 + |\mathbf{a}^e|\sin\theta_i}{2\kappa}\right)(y - y_{r,i})}$$

“Enrichment Function Bank”

Polynomial Family

$$c^E_{e,0}(\mathbf{x}) = 1$$
$$c^E_{e,1}(\mathbf{x}) = |\mathbf{a}^e \times \mathbf{x}|$$
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$$\vdots$$

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“Higher Order” Enrichment

$$c^E(\mathbf{x}) = \int_0^{\mathbf{v}_i \cdot \mathbf{x}} \exp\left\{\frac{\sigma_i w^2}{2} + (\mathbf{v}_i \cdot \mathbf{b}) w\right\} dw$$

$$\mathcal{V}^E_e$$

Element Nomenclature

Notation

DGM Element: $\begin{cases} Q-(n^{\text{pol}},n^{\text{exp}})-n^{\lambda} \\ Q-(n^{\text{pol}},n^{\text{exp}})^{*}-n^{\lambda} \end{cases}$

DEM Element: $Q-n^{\text{exp}}-n^{\lambda+} \equiv [Q-n^{\text{exp}}-n^{\lambda}] \cup [Q_1]$

$'Q'$: Quadrilateral
$n^{\text{exp}}/n^{\text{pol}}$: Number of Exponential/Polynomial Enrichment Functions
n^{λ}: Number of Lagrange Multipliers per Edge
Q_1: Galerkin Bilinear Quadrilateral Element

	Name	n^E	Θ_e^c	n^{λ}
DGM elements	$Q-(4,5)-2$	9	$\phi_e+\left\{\frac{2m\pi}{5}: m=0,...,4\right\}$	2
	$Q-(4,5)^{*}-2$	10	$\phi_e+\left\{\frac{2m\pi}{5}: m=0,...,4\right\}$	2
	$Q-(4,9)-3$	13	$\phi_e+\left\{\frac{2m\pi}{9}: m=0,...,8\right\}$	3
	$Q-(4,9)^{*}-3$	14	$\phi_e+\left\{\frac{2m\pi}{9}: m=0,...,8\right\}$	3
DEM elements	$Q-5-1^{+}$	5	$\phi_e+\left\{\frac{2m\pi}{5}: m=0,...,4\right\}$	1
	$Q-9-2^{+}$	9	$\phi_e+\left\{\frac{2m\pi}{9}: m=0,...,8\right\}$	2
	$Q-13-3^{+}$	13	$\phi_e+\left\{\frac{2m\pi}{13}: m=0,...,12\right\}$	3
	$Q-17-4^{+}$	17	$\phi_e+\left\{\frac{2m\pi}{17}: m=0,...,16\right\}$	4

Computational Complexities

Element	Asymptotic # of dofs	Stencil width for uniform $n \times n$ mesh	(# dofs) $\times$ (stencil width)	L^2 convergence rate (*a posteriori*)
Q_1	n_{el}	9	$9n_{el}$	2
Q_2	$3n_{el}$	21	$63n_{el}$	3
$Q-(4,5)-2$	$4n_{el}$	14	$56n_{el}$	3
$Q-(4,5)^*-2$	$4n_{el}$	14	$56n_{el}$	3
$Q-5-1^+$	$3n_{el}$	21	$63n_{el}$	$2-3$
Q_3	$5n_{el}$	33	$165n_{el}$	4
$Q-(4,9)-3$	$6n_{el}$	21	$126n_{el}$	4
$Q-(4,9)^*-3$	$6n_{el}$	21	$126n_{el}$	4
$Q-9-2^+$	$5n_{el}$	33	$165n_{el}$	$3-4$
Q_4	$7n_{el}$	45	$315n_{el}$	5
$Q-13-3^+$	$7n_{el}$	45	$315n_{el}$	$4-5$
$Q-17-4^+$	$9n_{el}$	57	$513n_{el}$	$4-5$

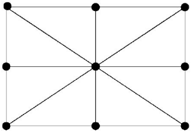


Figure 4: Q_1 stencil

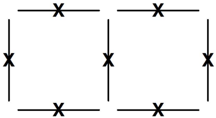


Figure 5: $Q-4-1$ stencil

Summary of Computational Properties

"COMPARABLES"

A priori **in computational cost:**
- DGM with n LMs and Q_n
- DEM with n LMs and Q_{n+1}

A posteriori **in convergence rate:**
- DGM with n LMs and Q_n
- DEM with n LMs and Q_n/Q_{n+1}

- Exponential enrichments $\Rightarrow$ integrations can be computed analytically.
- When $\mathbf{a}$ constant, $\mathcal{L}c^E = 0 \Rightarrow$ convert volume integrals to boundary integrals:

$$\begin{aligned} a(v^E, c^E) &= \int_{\tilde{\Omega}} (\kappa \nabla v^E \cdot \nabla c^E + \mathbf{a} \cdot \nabla c^E v^E)\, d\Omega \\ &= \int_{\tilde{\Gamma}} \nabla c^E \cdot \mathbf{n} v^E d\Gamma \end{aligned}$$

Inhomogeneous Rotating Advection Problem on an L–Shaped Domain

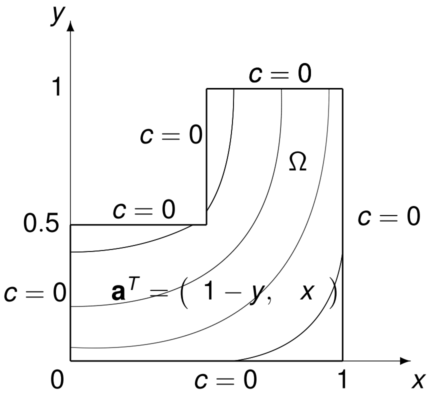
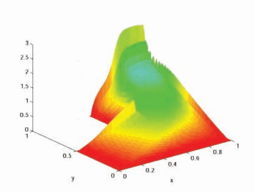


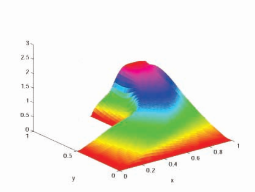
Figure 6: L-shaped domain and rotating velocity field (curved lines indicate streamlines)

- Homogeneous Dirichlet boundary conditions are prescribed on all six sides of L–shaped domain Ω.
- Source: $f = 1$.
- $\mathbf{a}^T(\mathbf{x}) = (\; 1 - y, \quad x \;)$.
- Outflow boundary layer along the line $y = 1$.
- Second boundary layer that terminates in the vicinity of the re-entrant corner $(x, y) = (0.5, 0.5)$.

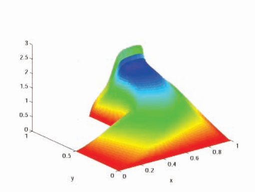
Solutions Plots for $Pe = 10^3$ with ≈ 3000 dofs



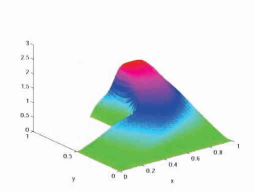
Q_1



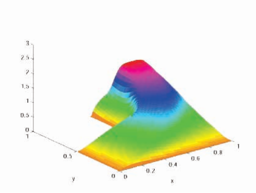
Stabilized Q_1



Q_2



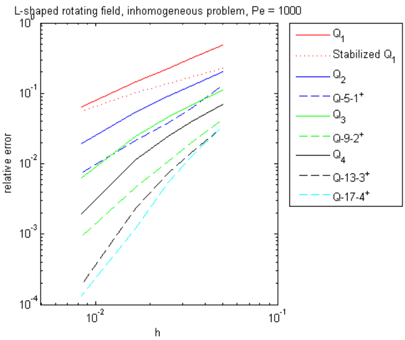
$Q-5-1^+$



$Q-9-2^+$

* "Stabilized Q_1" is upwind stabilized bilinear finite element [5].

Convergence Analysis & Results

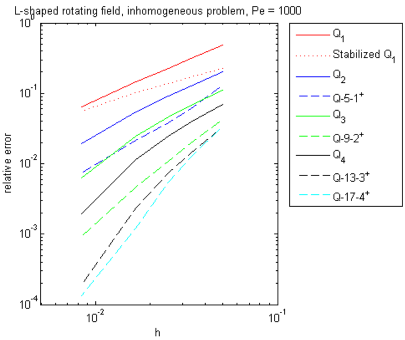


Element	Rate of convergence	# dofs to achieve 10^{-2} error
Q_2	1.94	62,721
$Q-5-1^+$	1.55	21,834
Q_3	2.67	33,707
$Q-9-2^+$	2.37	7,568
Q_4	3.50	20,796
$Q-13-3^+$	3.23	5,935
$Q-17-4^+$	3.26	4,802

* "Stabilized Q_1" is upwind stabilized bilinear finite element [5].

- To achieve for this problem the relative error of 1% for $Pe = 10^3$:
  - $Q-5-1^+$ requires 2.9 $\times$ **fewer** dofs than Q_2 (same **sparsity**).
  - $Q-9-2^+$ requires 4.5 $\times$ **fewer** dofs than Q_3 (same **sparsity**).
  - $Q-13-3^+$ requires 3.5 $\times$ **fewer** dofs than Q_4 (same **sparsity**).

Convergence Analysis & Results



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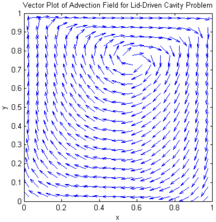
  - $Q-5-1^+$ requires 2.9 $\times$ **fewer** dofs than Q_2 (same **sparsity**).

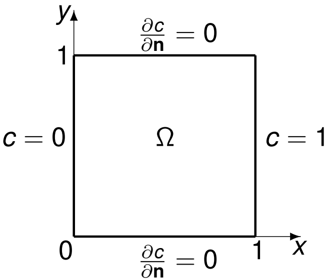
    $\Rightarrow$ 3.6 $\times$ *less CPU time.*
  - $Q-9-2^+$ requires 4.5 $\times$ **fewer** dofs than Q_3 (same **sparsity**).

    $\Rightarrow$ 9.2 $\times$ *less CPU time.*
  - $Q-13-3^+$ requires 3.5 $\times$ **fewer** dofs than Q_4 (same **sparsity**).

    $\Rightarrow$ 11.4 $\times$ *less CPU time.*

Lid-Driven Cavity Flow Problem



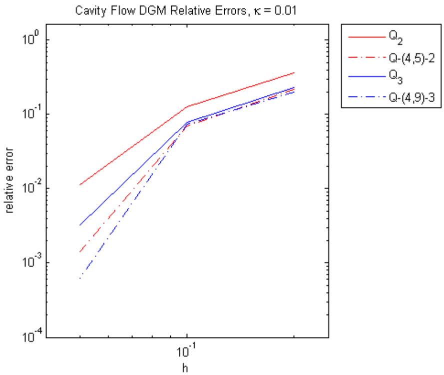


- $\Omega = (0,1) \times (0,1)$, $f = 0$.
- $\mathbf{a}(\mathbf{x})$ computed numerically by solving the incompressible Navier-Stokes equations for lid-driven cavity flow problem (stationary sides and bottom, tangential movement of top).
- Advection field reconstructed using interpolation with bilinear shape functions ϕ_i^e:

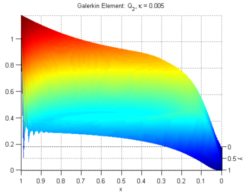
$$\mathbf{a}^e(\xi) = \sum_{i=1}^{\#\text{ nodes of }\Omega^e} \mathbf{a}_i^e \phi_i^e(\xi)$$

- $c(\mathbf{x})$ represents temperature in cavity.

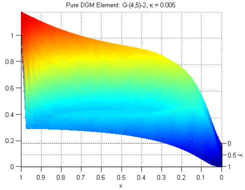
Convergence Analysis & Results ($\kappa = 0.01$, $Pe \approx 260$)



- Pure DGM elements without “higher order” enrichment outperform Galerkin comparables.

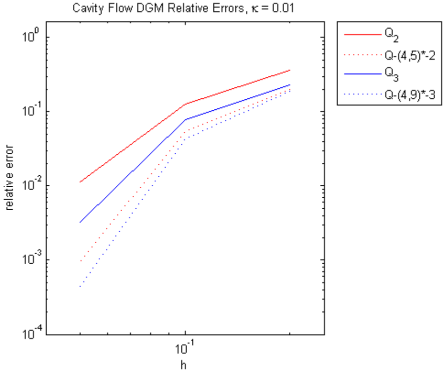


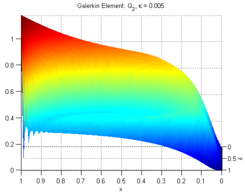
Q_2



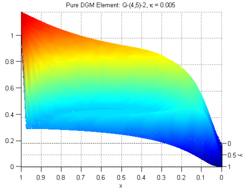
$Q-(4,5)-2$

Convergence Analysis & Results ($\kappa = 0.01$, $Pe \approx 260$)





Q_2



$Q - (4, 5) - 2$

- Pure DGM elements without "higher order" enrichment outperform Galerkin comparables.
- Further improvement in computation by adding "higher order" enrichment.

Summary

Discontinuous Enrichment Method (DEM) =
efficient, competitive alternative to stabilized FEMs
for advection-diffusion in a high Péclet regime.

- Parametrization of exponential basis enables systematic design of DEM elements of arbitrary orders.
- Augmentation of enrichment space with additional free-space solutions can improve further the approximation.
- For all test problems, enriched elements outperform their Galerkin and stabilized Galerkin counterparts of comparable computational complexity, sometimes by many orders of magnitude.
- In a high Péclet regime, DGM and DEM solutions are almost completely oscillation-free, in contrast with the Galerkin solutions.
- Advection-diffusion work generalizable to more complex equations in fluid mechanics (e.g., non-linear, unsteady, 3D).
- Future work: DEM for incompressible Navier-Stokes.

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